Determining the Angle and Axis of Rotation of a Resilient Modulus Specimen

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The angle of rotation is defined as the angle between he normal vector of the plane defined by the 3 LVDT deflections and a vertical vector.

Plane equation:

$$Ax + By + Cz + D = 0 \tag{1}$$

Angle between the normals of two planes[1]

$$\cos\tau = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \tag{2}$$

Plane passing through Three Points $\mathbf{P}_i$, $\mathbf{P}_j$, $\mathbf{P}_k$ (Tuma, p. 50)

$$\begin{vmatrix} y_i & z_i & 1 \\ y_j & z_j & 1 \\ y_k & z_k & 1 \end{vmatrix} x + \begin{vmatrix} z_i & x_i & 1 \\ z_j & x_j & 1 \\ z_k & x_k & 1 \end{vmatrix} y + \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} z = \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix} \tag{3}$$

Horizontal Plane Equation: z = 0

<u>LVDT readings</u>

Three points, 120° apart when projected in the X,Y plane:

$$\mathbf{P_1} = (R, 0, \delta_1) \tag{4}$$

$$\mathbf{P}_2 = \left(-\frac{R}{2}, \frac{R\sqrt{3}}{2}, \delta_2\right) \tag{5}$$

$$\mathbf{P}_3 = \left(-\frac{R}{2}, -\frac{R\sqrt{3}}{2}, \delta_3\right) \tag{6}$$

Where
R = radius of specimen
δ_i = peak deflection of LVDT_i

[1] Tuma, Jan J., <u>Engineering Mathematics Handbook, 3rd Ed.</u>, McGraw-Hill, Inc., 1987, p. 51.

Equation for LVDT Plane

Solving determinants in Equation 3 using coordinates from Equations 4, 5, and 6:

$$A = \sqrt{3}R\left(\frac{\delta_2}{2} + \frac{\delta_3}{2} - \delta_1\right) \tag{7}$$

$$B = \frac{3}{2}R\left(\delta_3 - \delta_2\right) \tag{8}$$

$$C = \frac{3\sqrt{3}}{2}R^2 \tag{9}$$

$$D = -\frac{\sqrt{3}}{2}R^2\left(\delta_1 + \delta_2 + \delta_3\right) \tag{10}$$

Angle of Rotation, θ (Angle between normal of LVDT plane and vertical)

Substituting Equations 7, 8, 9, and 10 into Equation 2:

$$\cos\theta = \frac{\frac{3}{2}R}{\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_1\delta_3 - \delta_2\delta_3 + \frac{9}{4}R^2}} \tag{11}$$

In terms of diameter, *D*:

$$\cos\theta = \frac{\frac{3}{4}D}{\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_1\delta_3 - \delta_2\delta_3 + \frac{9}{16}D^2}} \tag{12}$$

Axis of Rotation

The axis of rotation is the intersection of the LVDT plane with the horizontal plane:

$$z = \frac{\delta_1 + \delta_2 + \delta_3}{3} \tag{13}$$

The equation for the intersection of two planes in the X,Y plane is (Tuma, p. 51):

$$\begin{vmatrix} C_1 & C_2 \\ A_1 & A_2 \end{vmatrix} x + \begin{vmatrix} C_1 & C_2 \\ B_1 & B_2 \end{vmatrix} y + \begin{vmatrix} C_1 & C_2 \\ D_1 & D_2 \end{vmatrix} = 0 \tag{14}$$

Substituting Equations 7, 8, 9, 10, and 13 into Equation 14 results in the equation for the axis of rotation:

$$\sqrt{3}R\left(\frac{\delta_2}{2} + \frac{\delta_3}{2} - \delta_1\right)x + \frac{3}{2}R\left(\delta_3 - \delta_2\right)y = 0 \tag{15}$$